ERROR ANALYSIS OF THE FINITE VOLUME BASED REGIONAL SIMULATION MODEL, RSM

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A fully integrated regional simulation model (RSM) has been developed to simulate the complex hydrologic system of South Florida. The South Florida system consists of a large integrated overland flow system, a groundwater flow system and a canal flow system. Implicit solution methods, efficient sparse solvers, and object oriented methods have made it possible to solve large complex systems simultaneously with practically any time step and cell size, even when the numerical errors can be extremely large. The current study is aimed at understanding the numerical error due to a boundary disturbance under a variety of spatial and temporal discretizations and solution characteristics. The errors are compared with the analytical estimates obtained by Lal (2000) for finite difference problems. Results of the study are useful in determining optimal spatial and temporal discretizations for model applications.

INTRODUCTION

Numerical models give inaccurate results if not used correctly. Numerical models based on the solution of partial differential equations can produce accurate solutions only for a limited range of spatial and temporal discretizations. Outside this range, model results have an added uncertainty. Attempts to obtain extremely accurate solutions using very fine meshes is however prohibited due to the cost of the model run.

Numerical error can contribute to a significant portion of model

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uncertainty depending on the spatial and temporal discretization. In order to keep error under control, spatial and temporal discretizations of models have to be carefully selected. The current study is aimed at determining how the numerical error of a model relates to the size of the triangular cells, the time step length, and the characteristics of the solution in the case of the RSM model. Results of the study are presented in dimensionless form so that they can be used in the future model applications.

Numerical errors of groundwater flow models and diffusion type overland flow models with rectangular cells have been studied in the past by Lal (2000a) and (2000b). These studies focused on errors due to boundary disturbances, spatially varying rainfall patterns, transient pumping and stream-aquifer interaction. Errors due to boundary disturbances are investigated in the current study.

GOVERNING EQUATIONS

The governing equation for 2-D groundwater flow and overland flow with negligible inertia effects can be expressed as (Akan and Yen, 1981, Lal, 1998)

$$s_c \frac{\partial H}{\partial t} + \frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} + W = 0 \tag{1}$$

in which, u and v are the velocities in the x and y directions; H = water head; R = net contribution of recharge from local hydrology into the regional system; W = source or sink terms due to pumping wells, etc.; s_c = storage coefficient. $s_c = 1$ for overland flow. For both overland flow under diffusion flow assumption (Akan and Yen, 1981) and for groundwater flow,

$$u = -\frac{T}{d}\frac{\partial H}{\partial x}, \quad v = -\frac{T}{d}\frac{\partial H}{\partial y} \tag{2}$$

For groundwater flow, $T = T(H)$ = transmissivity of the aquifer as a function of the water level. For overland flow, $T = C(H)S^{\lambda-1}$ in which $C(H)$ is defined as the conveyance; S = water surface slope; λ = an empirical constant; d = water depth. Both $T(H)$ and $C(H)$ are useful in describing a whole range of overland and groundwater flow behaviors.

The implicit finite volume method used to solve the partial differential equations is explained in detail by Lal (1998). Even if the mass balance errors in the finite volume methods are small, computational errors are present in all numerical models including finite volume models because of errors of discretization. Following experiments are conducted to determine this error.

NUMERICAL EXPERIMENTS

Unlike in the case of rectangular cells, a variety of cell sizes and shapes are possible with finite volume models using triangular cells. In order to study the error behavior, a confined ground water model was set up over a 10 $km \times 10$ km square domain using a triangular mesh of 3200 approximately isometric triangles. The software package GMS was used to generated a mesh of approximately isometric triangles of approximately the same size. Figure 1 shows part of the exact mesh used in the experiment. The spatial discretization is made dimensionless using the equation

$$\phi = k\sqrt{\Delta A} \tag{3}$$

in which $k =$ wave number of the disturbance assuming isotropy; $\Delta A =$ cell area. With rectangular cells, the definition reduces to $\phi = k\Delta x$.

In the experiment, a one dimensional boundary disturbance was introduced into the domain by changing the water head of one of the boundary walls in a sinusoidal manner. Water heads at different distances away from the wall were measured over long periods and compared with their analytical estimate to obtain the measured errors in the amplitude. These errors were then compared with analytical error estimates assuming a uniform mesh (Lal, 2000). The analytical expression for the head is

$$H(x,t) = H_o e^{-kx} \sin(ft - kx) \tag{4}$$

in which, $f =$ frequency of the boundary disturbance in radians per second; $H_o =$ amplitude of the disturbance; $x =$ distance away from the boundary; $k = \sqrt{fs_c/(2T)}$. The analytical expression for maximum numerical error assuming a uniform mesh is (Lal, 2000).

$$\varepsilon_T(x) = \frac{k\varepsilon}{\beta\phi^2}x = \frac{2k\varepsilon}{\psi}x \tag{5}$$

in which, $\varepsilon_T(x) =$ maximum error in the amplitude at a distance x; $\psi = f\Delta t =$ dimensionless time step; $\Delta t =$ time step; $\varepsilon =$ maximum error per time step as a fraction of the amplitude; $\beta = K\Delta t/(s_c \Delta A)$; and $\beta = \psi/(2\phi^2)$ for the boundary disturbance problem. Values of ε can be obtained using algebraic equations presented by Lal (2000).

The first numerical experiment was carried out to compare numerical errors of an actual model run with the numerical errors computed using the analytical equation (5). Part of the mesh used in the numerical test bed is shown in Fig. 1. A confined aquifer of transmissivity of 2.0 m^2/s was used in the test bed. As the wall head boundary condition, a water head varying sinusoidally with an amplitude of 100 m and a period of 20 hours was used. In the model simulation, $\Delta t = 1$ day was used. The mesh was used to obtain the average $\sqrt{\Delta A}$ as approximately 152 m. Figure 2 shows the plot of the amplitude with the distance from the wall boundary. The analytical amplitudes are shown with a continuous line. Table 1 also shows the amplitudes obtained with the RSM model. To obtain the analytical solution, $\phi = 0.06481$, $\psi = 0.31459$, $\beta = 37.396$, and $\varepsilon = 0.011291$ (from Lal, 2000) was used. Both the figure and the table show how the absolute error increase and decrease with distance from the wall that is disturbed. Figure 2 shows that the maximum numerical error of the RSM model can be explained using the analytical expression. Figure 3 also shows that variation of the numerical error as a fraction of the amplitude varies as a straight line with the distance.

Since actual model error as a fraction of the amplitude varies linearly with distance from the boundary as shown in Fig. 3, it becomes possible to use its slope to obtain the value of ε using (5). This value can be compared with the analytical estimate of ε. In Fig. 3, the slope computed using least square best is 3.05274×10^{-5}. Equation (5) can be used with $k = 0.0004264$, and $\psi = 0.31459$ to obtain ε as 0.011246. This agrees with ε computed analytically as 0.011291.

The maximum analytical error per time step ε for any spatial discretization ψ and temporal discretization ϕ can be made into a contour plot as in Figure 4. The value of ε obtained from just one model run of the RSM can also be plotted as a dot on the same figure. Values of ε are useful in determining numerical errors under a variety of situations. If the error is too large for given discretizations, the discretizations have to

Table 1: Detatils of one experimental model run

cell ID	distance x (m)	model ampl. y_{obs}, m	anal. ampl. y_{ana}, m	error ε_a, (m)	$\varepsilon_T = \varepsilon_a/y_{ana}$
wall	0.0	100.00	100.00	0.000	0.0000
1179	67.3	96.93	97.17	0.243	0.0025
1181	271.1	87.89	89.08	1.197	0.0134
1182	345.3	85.07	86.31	1.236	0.0143
1302	737.8	72.10	73.01	0.911	0.0125
1424	1252.5	56.00	58.62	2.625	0.0448
1921	2451.8	32.62	35.15	2.534	0.0721
2225	4961.8	10.20	12.05	1.852	0.1537

be changed. Errors in many types of 1-D and 2-D problems are listed in the paper by Lal (2000).

The results of the model runs can be used also to verify the way in which the maximum error due to a boundary disturbance behaves spatially. In order to obtain the analytical solution to this problem, (4) and (5) can be used to obtain the maximum absolute error ε_a as $H(x,t)\varepsilon_T(x)$.

$$\varepsilon_a = H_o e^{-kx}\frac{2k\varepsilon}{\psi}x \tag{6}$$

Figure 2 shows this behavior in which the the absolute error increases and then decreases with distance from the boundary. The maximum of this function can be obtained by differentiating (6).

$$\varepsilon_{max} = \frac{2H_o\varepsilon}{e\psi} \tag{7}$$

or approximately $\varepsilon_{max} = 0.736H_o\frac{\varepsilon}{\psi}$ which takes place at a distance $x_{max} = 1/k$ from the boundary. When applied to the numerical experiment with the RSM, $k = 0.0004264$, and therefore $x_{max} = 2345.3$ m for the error to be maximum. The maximum of the maximum error $\varepsilon_{max} = 0.736H_0\varepsilon/\psi$ $= 0.736 * 100 * 0.0112911/0.314159$ = 2.644 m. It has been shown earlier (Lal, 2000) that the error associated with the MODFLOW model (McDonald and Harbough, 1988) behaves in the same way.

SUMMARY AND CONCLUSIONS

Numerical error analysis of the Regional Simulation Model (RSM) was carried out by applying a sinusoidal water head fluctuation to the boundary of a simple model setup. The model error was computed using an analytical solution developed by Lal (2000) as the exact solution. In applying the method, the spatial discretization of triangles was measured as the square root of the cell area. Results of the study show that the maximum numerical error as a fraction of amplitude varies linearly with the distance from the boundary as predicted in the analytical derivation. Results also show that the numerical error of RSM agree closely with the analytical estimates. These results are extremely useful in model development because they verify that the numerical methods are properly applied. If the model errors are larger or smaller, there is a problem with the implementation of the numerical method.

Results of error analysis can be used in designing the cell size and the time step length. If the numerical error is large, the discretization has to be nade smaller intil the required solution can be properly represented.

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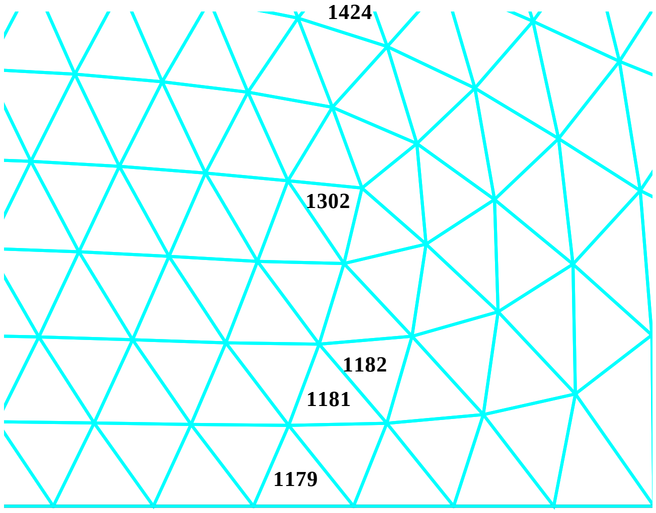

Figure 1: Part of the mesh used for the numerical experiment

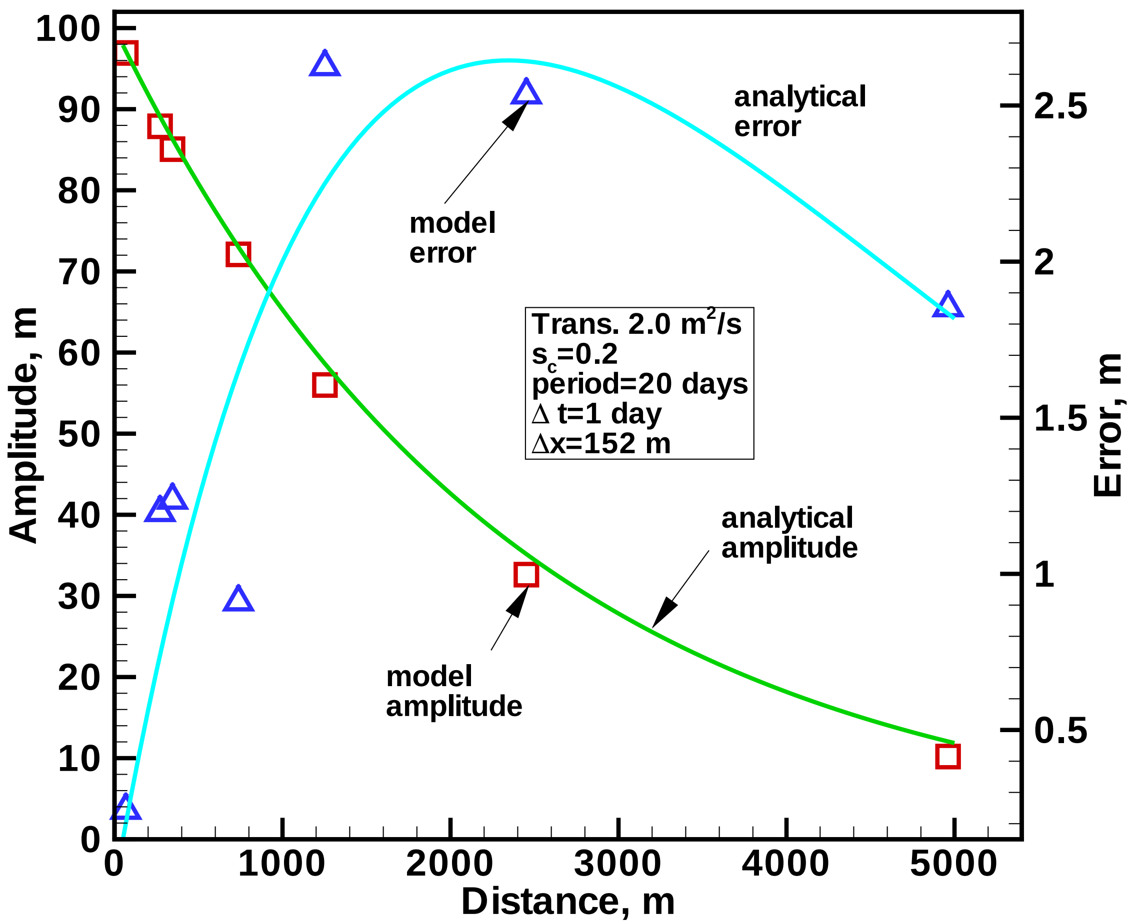

Figure 2: Variation of the amplitude and the maximum absolute numerical error with distance

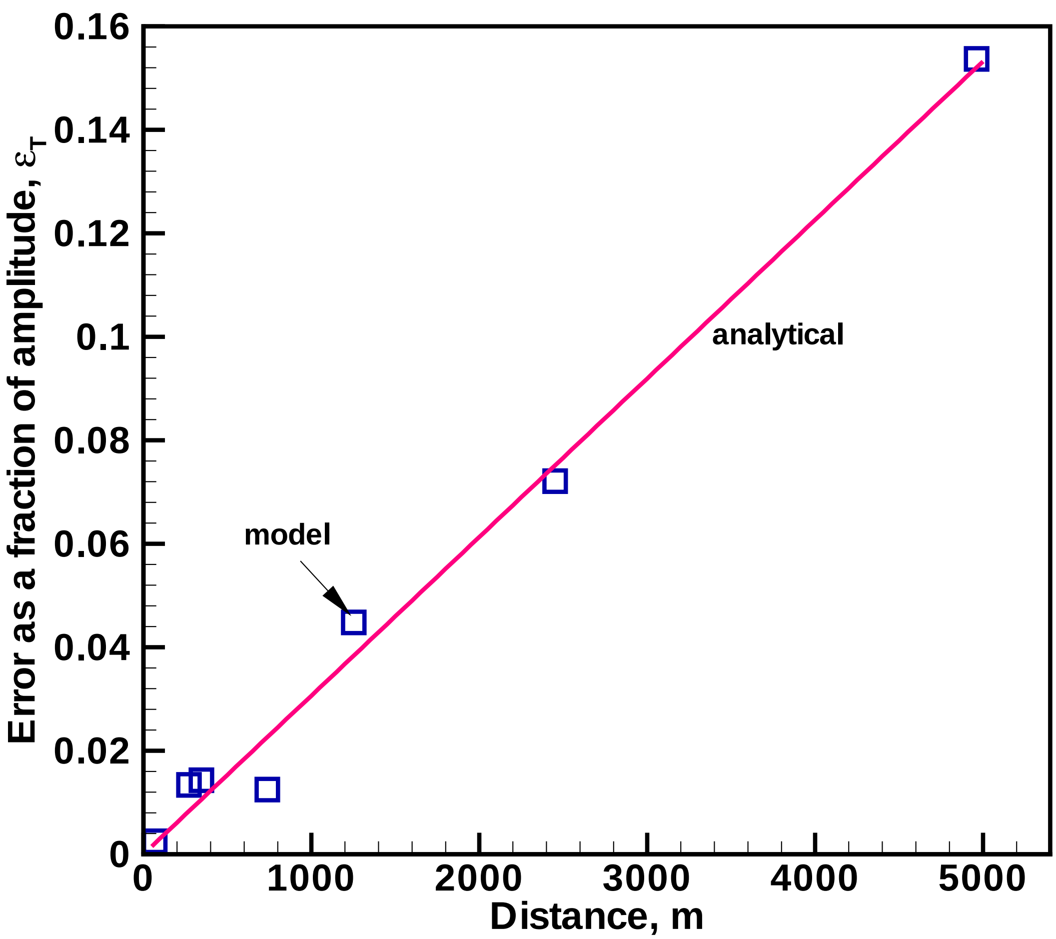

Figure 3: Variation of maximum numerical error as a fraction of the amplitude with distance

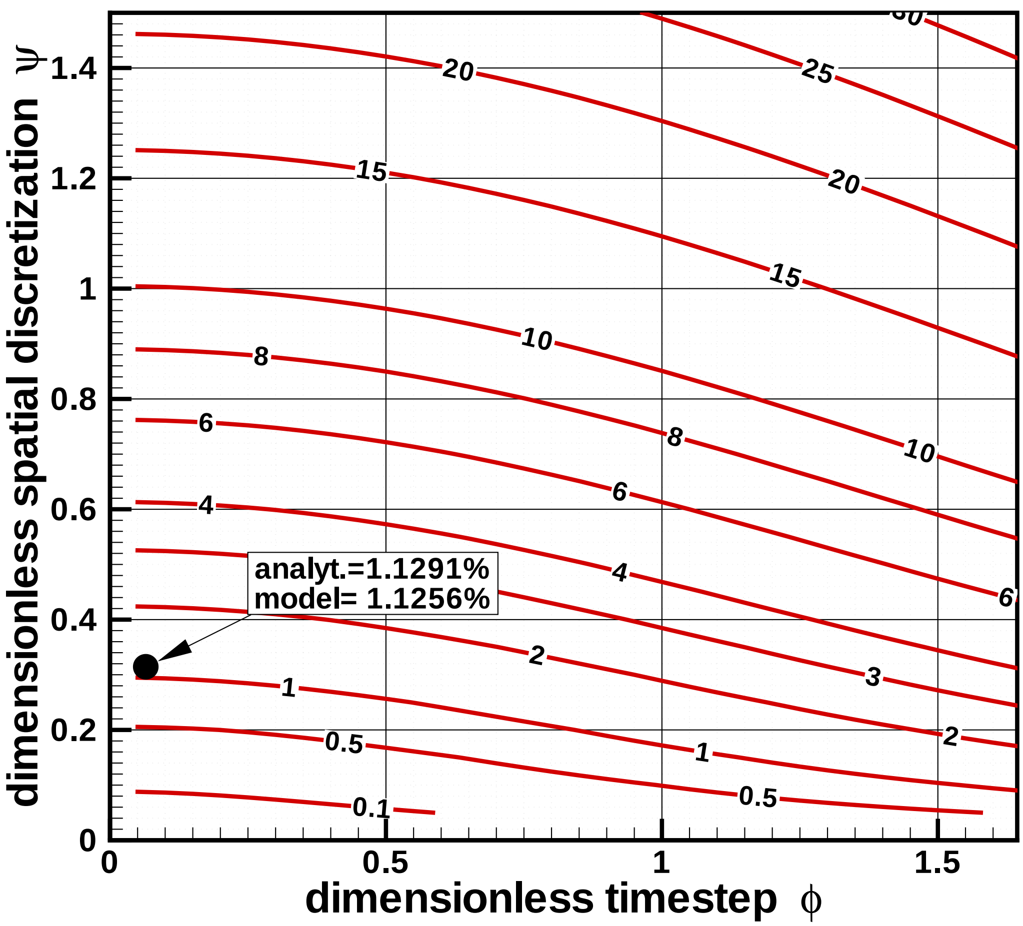

Figure 4: Contours of maximum numerical error per time step ε (%) against dimensionless spatial and temporal discretizations φ and ψ